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$$\frac{d}{dx}f(x) \quad \sum_{k=0}^{+\infty} a_k \quad \int f(x)\,dx \quad \oint_{\Gamma} (X dx + Y dy + Z dz)$$

Predictive Mathematical Models of the Covid–19 pandemic in ODE/SDE framework

Marcello Colozzo

Abstract

This article proposes a viral diffusion model (like Covid-19 pandemic) in the *ordinary differential equations* (**ODE**) and *stochastic differential equations* (**SDE**) framework. The classic models based on the logistic map are analyzed, and then a noise term is introduced that models the behavior of the so-called *deniers*. This model fairly faithfully reproduces the Italian situation in today's period. We then move on to local analysis, arriving at an equation of continuity for what concerns the density of the number of infected in an assigned region. We, therefore, prove a Theorem according to which classical logistics is the most catastrophic of predictions. In a realistic scenario, it is necessary to take into account the inevitable fluctuations in the aforementioned density. This implies a fragmentation of the initial cluster (generated by "patient zero") into an N disjoint sub clusters. For very large N, statistical analysis suggests the use of the *two-point correlation function* (and more generally, n-points). In principle, an estimate of this function makes it possible to determine the evolution of the pandemic. The distribution of the sub clusters could be fractal, exactly as it happens for the distribution of galaxies starting from a homogeneous and isotropic primordial universe, but with random fluctuations in matter density. This is not surprising, since due to the invariance in scale, fractals have a low "computational cost". The idea that pandemics are cyclical processes, that is, they occur with a given periodicity, would therefore remain corroborated.

Keywords: pandemic, wiener process,stochastic differential equation

Contents

Introduction

The work is divided as follows:

- In sections 1, 2, 2.1 the necessary definitions are introduced, and then set up a differential equation containing among its coefficients a containment parameter (lockdown, social distancing, masks, etc.).

- Section 2.2 considers the special case of an autonomous system which is specifically a Bernoulli-like differential equation with constant coefficients (stationary containment action). By applying Fourier's analysis to the density fluctuations of the number of infected, we prove a Theorem according to which the logistic solution is the most catastrophic of the predictions.

- In section A.4 the use of the *two-point correlation function* (or more generally n-points) is proposed, the estimate of which can be made by analyzing the data collected. In this way it could be possible to arrive at a law of distribution of infected clusters, starting from an initial homogeneous cluster. A fractal distribution of the aforementioned clusters cannot be excluded.

- In section 2.3 we move on to a more realistic scenario where the containment parameter depends on time. We prove a Theorem according to which a pandemic modeled by a non-autonomous system with a containment parameter represented by an analytic function, is extinguished at most asymptotically.

- Section 2.4 mentions an alternative paradigm to that of differential equations, with particular regard to autonomous systems. More precisely, the dynamic evolution of the pandemic is studied in the space of configurations.

- Section 3 considers an even more realistic scenario, in which the containment parameter "disturbed" by the action of the so-called *deniers*. The disturbing action is modeled by a Wiener process.

1 Predictability and controllability of a pandemic

If t_0 and t_{n_a} denote the initial and current instants of a pandemic process $\mathcal{P}$ respectively, the time interval $[t_0, t_{n_a}]$ is sampled in intervals of width $\Delta = 1\,\mathrm{d}$. If η_k is the number of daily infections, i.e. the number of cases registered on the *k-th* day t_k (new positives), the following subset of $\mathbb{N}$ ($n_a > 1$) is uniquely determined:

$$\{\eta_0, \eta_1, ..., \eta_{n_a}\} \tag{1}$$

As n_a increases indefinitely, la (1) becomes a sequence from $\mathbb{N}$ to $\mathbb{N}$:

$$\{\eta_k\} : \eta_0, \eta_1, \eta_2, ... \tag{2}$$

That said, the following definitions exist:

Definition 1 *A pandemic $\mathcal{P}$ è* **predictable** *if (2) is elementarily expressible. In otherwise we say that $\mathcal{P}$ is* **unpredictable**.

Definition 2

$$\Sigma_{\mathcal{P}} \stackrel{def}{=} \{\sigma_1, \sigma_2, ..., \sigma_p\} \tag{3}$$

where σ_k denotes a containment action (social distancing, masks, lockdown, antiviral therapies, vaccine). An unpredictable pandemic $\mathcal{P}$ is **controllable** *if*

$$\exists \Sigma_{\mathcal{P}} \neq \emptyset \mid \eta_k = 0, \quad \forall k > k_* \in \mathbb{N} \tag{4}$$

Otherwise, $\mathcal{P}$ is said ti be **uncontrollable**.

2 Attack strategies

- Time domain [1] (ODE, PDE, SDE)

  Notation 3 *As we will see below, the use of SDE derives from the presence of random variables. More specifically, if $\mathcal{I}$ is a population of individuals, it follows that however we take $I \in \mathcal{I}$, I is free to make choices and therefore to execute a behavior conforming to at least one $\sigma_k \in \Sigma_{\mathcal{P}}$.*

- Configuration domain [2] (CA)

2.1 Time domain

In the hypothesis of predictability of $\mathcal{P}$, we are obviously interested in the search for the expression elementary of succession (2).). For this purpose it is preferable to pass to the continuum, defining first the magnitude y_k:

$$y_{k+1} - y_k = \eta_k, \quad k = 0, 1, 2, ..., n_a - 1 \tag{5}$$

which enumerates the total number (at the current time t_{n_a}) of the infected (the so-called *currently positive*). Equation (5) can be rewritten as

$$\frac{y_{k+1} - y_k}{\Delta} = \eta_k \tag{6}$$

Going to the continuous and performing the operation of passing the limit for $\Delta \to 0$, the previus become:

$$\frac{d}{dt}y(t) = \eta(t)$$

Using dotted notation to denote the derivation operation with respect to time:

$$\dot{y}(t) = \eta(t) \tag{7}$$

Since it is simpler to refer to the magnitude $y(t)$ which enumerates the currently positive, any predictive model will have to implement a differential equation for the aforementioned greatness. In order to define the characteristic parameters of a pandemic, let us examine the special case given by the following homogeneous linear ODE:

$$\dot{y} = R_0 y \tag{8}$$

being $R_0 > 0$ a constant with the dimensions of the inverse of a time. Equation (8) is accompanied by the initial condition:

$$y(t_0) = y_0, \tag{9}$$

The solution of this Cauchy problem is:

$$y(t) = y_0 e^{t/\tau}, \tag{10}$$

were $\tau = R_0^{-1}$ is the *time constant* of the exponential viral diffusion process. For being able to explain the meaning of $R_0 = \frac{\dot{y}}{y}$ we have to sample to sample time, so that with obvious meaning of symbols

$$R_0 = \frac{\frac{y_{k+1} - y_k}{\Delta}}{y_k}, \quad (\Delta = 1\,\mathrm{d}) \tag{11}$$

For example for $y_k = 1$ we have that for $y_{k+1} > y_k$, it succeeds

$$R_0 \underset{\Delta = 1\,\mathrm{d}}{=} y_{k+1} > 1 \tag{12}$$

This implies that if at the instant t_k we have only one infected, at the next instant $t_{k+1} = t_k + \Delta$ there will be $R_0 > 1$ iinfected. From an epidemiological point of view, this means that on average a infected can infect R_0 individuals in the time interval Δ.

Definition 4 *The constant R_0 that appears in (8) is called* **the contagio rate**.

In the paradigm of systems theory, (8) describes a *linear dynamic system*. More precisely, (8) is a particular case of a so-called *autonomous system*

$$\begin{cases} \dot{y} = f(y) \\ y(t_0) = y_0 \end{cases}, \tag{13}$$

being $f(y)$ an assigned function sufficiently regular in order to guarantee its existence and the uniqueness of the solutions of the Cauchy problem (13). The differential equation

$$\dot{y} = f(y) \tag{14}$$

integrates by separation of variables. The general integral is written:

$$F(y) = t + C, \quad \forall C \in \mathbb{R}$$

where is it

$$F(y) = \int \frac{dy}{f(y)}$$

The following geometric locus is uniquely determined in the Cartesian plane $(y, \dot{y})$:

$$\Gamma(f) = \left\{(y, \dot{y}) \in \mathbb{R}^2 \mid 0 \leq y < +\infty, \ \dot{y} = f(y)\right\} \tag{15}$$

that is the cartesian diagram of the real function f of the real variable y. For the aforementioned regularity hypothesis of f, we have that the locus $\Gamma(f)$ is a regolar curve.

Let's consider a non-autonomous system:

$$\dot{y} = f(t, y),$$

for an assigned real function f of the real variables t, y, sufficiently regular so as to ensure the existence and uniqueness of the solutions for a given initial condition $y(t_0) = y_0$. Virtually, the function $f(t, y)$ is expressed as the sum of two contributions:

$$f(t, y) = f_{auto}(y) + f_{cont}(t, y), \tag{16}$$

where the first term in the second member is clearly the contribution coming from the dynamics "internal" to the system. The second term, on the other hand, represents the containment action (3). The latter explicitly depends on time through a term we denote with $\beta(t)$:

$$f_{cont}(t, y) = \beta(t)\, y^{\lambda},$$

being $\lambda > 0$ a parameter that introduces a nonlinear effect, essential if we want maintain an adherence to physical reality (most of the processes occurring in nature are nonlinear, so linearity is only a useful approximation).

Definition 5 *The quantity $\beta(t)$ is called the* **pandemic containment parameter***.*

Definition 6 *If $y(t)$ is the only solution of the Cauchy problem*

$$\begin{cases} \dot{y} = f(t, y) \\ y(t_0) = y_0 \end{cases},$$

the greatness

$$R(t) \overset{def}{=} \frac{\dot{y}(t)}{y(t)} \tag{17}$$

is called the **actual contagion rate***.*

Remark 7 *A pandemic $\mathcal{P}$ is controllable if*

$$\exists t_1 > 0 \mid R(t) < 1, \quad \forall t > t_1; \tag{18}$$

not controllable otherwise.

2.2 Autonomous systems. Solution analysis

At this point it is interesting to resume autonomous systems, examining the special case of following stationary containment process ($\beta(t) \equiv \beta_0 =$ constant):

$$f_{cont}(y) = \beta_0, \quad \lambda = 2$$

hence our Cauchy problem is rewritten:

$$\mathcal{C} : \begin{cases} \dot{y} = R_0 y - \beta_0 y^2 \\ y(t_0) = y_0 \end{cases} \tag{19}$$

Here $R_0 > 1$ and $\beta_0 \geq 0$ such that

$$R_0 y(t) - \beta_0 y(t)^2 > 0, \quad \forall t \in [t_0, +\infty) \tag{20}$$

whereby $\dot{y} > 0$, i.e. the function $y(t)$ is monotonically increasing. This condition is essential since the function $y(t)$ expresses the total number of infected persons at time t, counted from the initial instant t_0. Since $y(t)$ is zerozero, (20) can be rewritten:

$$0 \leq \beta_0 < \frac{R_0}{y(t)}, \quad \forall t \in [t_0, +\infty)$$

Excluding the trivial case $\beta_0 = 0$, we have that the differential equation (19) is Bernoulli-like, and the solution of $\mathcal{C}$ is:

$$y(t) = \frac{L_0}{1 + \left(\frac{L_0}{y_0} - 1\right) e^{-R_0 t}}, \tag{21}$$

having defined:

$$L_0 = \frac{R_0}{\beta_0} = \lim_{t \to +\infty} y(t) \tag{22}$$

This magnitude is the total number of people infected at the end of the pandemic. For example, if $R_0 = 2.2$, $\beta_0 = 0.2$, we find the trend shown in fig. 1. . Daily contagions are enumerated by the first derivative of function(21), whose graph has the trend shown in fig. 2. The inflection point of the graph of $y(t)$ corresponds to a relative maximum point corrisponde of the derivative $\dot{y}(t)$, technically known as the *maximum peak* (fig. 3).

The actual contagion rate is

$$R(t) = \frac{\dot{y}(t)}{y(t)} = R_0 \frac{\left(\frac{L_0}{y_0} - 1\right) e^{-R_0 t}}{1 + \left(\frac{L_0}{y_0} - 1\right) e^{-R_0 t}},$$

graphed in fig. 4.

From (22) we see that the pandemic described by the autonomous system (19) is not controllable at the finite, but it is asymptotically since the daily contagions cancel each other out only by $t \to +\infty$. To be more practical, we assign an $\varepsilon \sim 1$ and then assume for the *instant of stopping* pandemic the time t_* such that

$$L_0 - y(t_*) < \varepsilon \tag{23}$$

The aforementioned instant uniquely defines a single *pandemic cycle*. Nothing prevents us from considering a second cycle (*second wave*) as a solution to a Cauchy problem analogous to problem (19) in which the initial instant is t_*, and the corresponding solution comes suitably connected with the previous one.

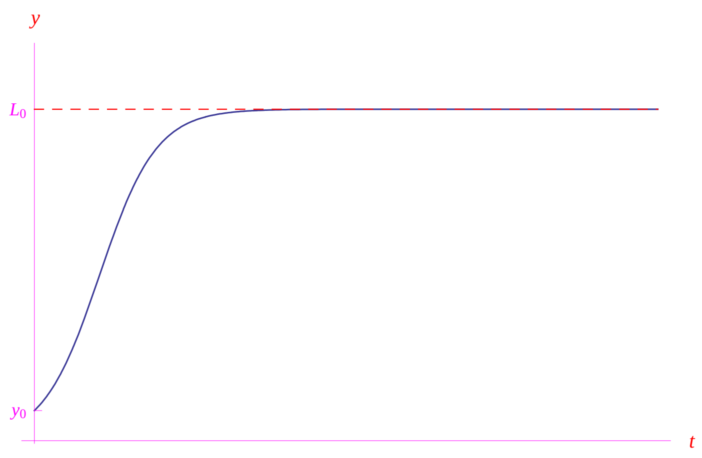

Figure 1: Trend of the solution of problem (19) for $R_0 = 2.2, \beta_0 = 0.2$. Note the asymptotic behavior.

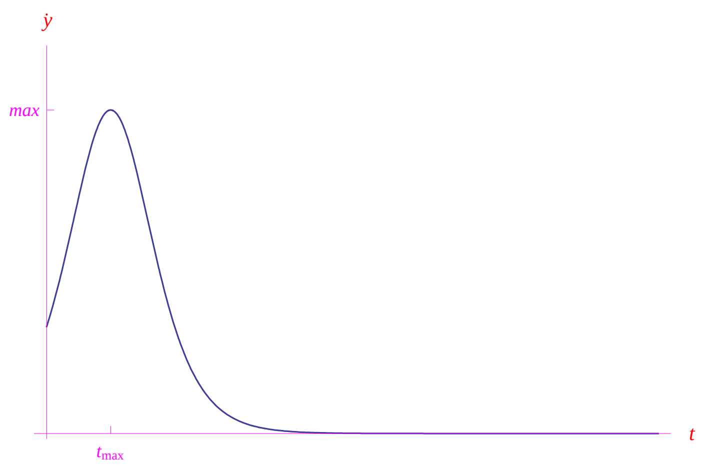

Figure 2: Trend of the first derivative of the solution of problem (19) for $R_0 = 2.2, \beta_0 = 0.2$.

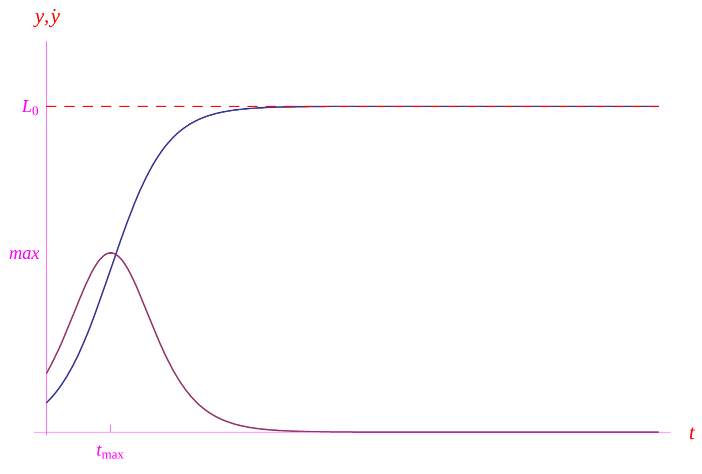

Figure 3: Trend of the solution and its first derivative of the solution of problem (19) for $R_0 = 2.2, \beta_0 = 0.2$.

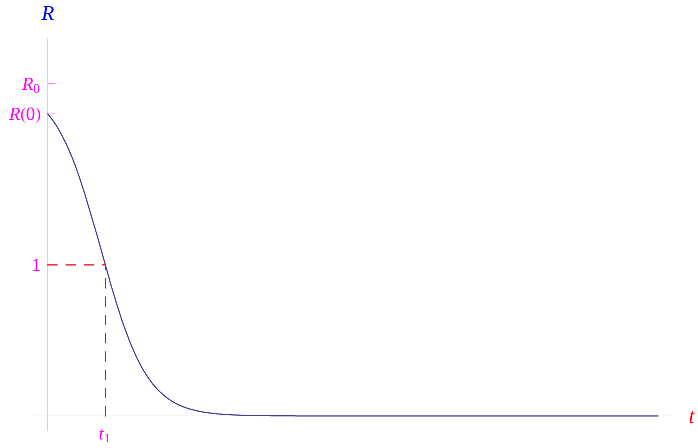

Figure 4: Trend of the actual infection rate as a function of time.

Definition 8 *By varying the parameters R_0, β_0 in (19) we obtain a family $\mathcal{F}$ of curves integrals, which solve infinite Cauchy problems sharing the same initial condition $y(0) = y_0$. Each element of this family is known as a* **logistic curve** *or simply* **logistic**.

Theorem 9 *For a controllable pandemic, logistics is the most catastrophic of predictions.*

Proof. See Appendix A. ■

2.3 Non-autonomous systems. Solution analysis

We generalize the behavior discussed in the previous section (§ 2.2) to a system not autonomous[1]:

$$\begin{cases} \dot{y} = R_0 y - \beta(t) y^2 \\ y(0) = y_0 \end{cases}, \tag{24}$$

taking on

$$\beta \in C^2([0, +\infty))$$
$$\beta(t) > 0, \quad \forall t \in [0, +\infty)$$

As usual we have to impose $\dot{y} > 0$, so we are interested in the solutions $y(t)$ of the Cauchy problem (24) such that

$$y_0 \leq y(t) < \frac{R_0}{\beta(t)} \tag{25}$$

For known theorems on limits:

$$y_0 \leq y(t) < \frac{R_0}{\beta(t)} \Longrightarrow y_0 \leq \lim_{t \to +\infty} y(t) \leq R_0 \lim_{t \to +\infty} \frac{1}{\beta(t)} \tag{26}$$

If the function $\beta(t)$ is infinitesimal for $t \to +\infty$

$$\lim_{t \to +\infty} y(t) \leq +\infty$$

That is, the solution $y(t)$ can be convergent or divergent for $t \to +\infty$. However, in a neighborhood of $+\infty$ the solutions of (24) (assuming that $\beta(t)$ is infinitesimal to infinity), we behave like the solutions of the differential equation $\dot{y} = R_0 y$, so

$$y(t) \underset{t \to +\infty}{\longrightarrow} e^{R_0 t} \Longrightarrow \lim_{t \to +\infty} y(t) = +\infty \tag{27}$$

[1]For simplicity we set $t_0 = 0$.

Conclusion 10 *If the function $\beta(t)$ is infinitesimal for $t \to +\infty$, the pandemic described by (24) is not controllable.*

If $\beta(t)$ is not infinitesimal to infinity, the possible behaviors are:

- $\beta(t)$ is oscillating for $t \to +\infty$, so

$$\nexists \lim_{t \to +\infty} \beta(t)$$

- $\beta(t)$ is convergent for $t \to +\infty$

$$\lim_{t \to +\infty} \beta(t) = \ell \in \mathbb{R}$$

In the case of convergence, the dynamic system is asymptotically autonomous, so yes expect a "logistical" trend.

To be more quantitative, the only solution to the Cauchy problem (24) (see. Appendix B) is

$$y(t) = \frac{y_0 e^{R_0 t}}{1 - y_0 [B_0 - B(t)]}, \tag{28}$$

where is it

$$B(t) \stackrel{def}{=} \int \beta(t) e^{R_0 t} dt, \quad B_0 = B(0) \tag{29}$$

Notation 11 *The function $B(t)$ does not contain the integration constant, since the latter is incorporated in (85).*

Deriving:

$$\dot{y} = \frac{y_0 e^{R_0 t} \left\{ R_0 [1 - y_0 (B_0 - B(t))] - y_0 \beta(t) e^{R_0 t} \right\}}{[1 - y_0 [B_0 - B(t)]]^2} \tag{30}$$

For the above, the first derivative $\dot{y}$ of the function that enumerates the total contagions is positive in $[0, +\infty)$, so

$$R_0 [1 - y_0 (B_0 - B(t))] - y_0 \beta(t) e^{R_0 t} > 0, \quad \forall t \in [0, +\infty)$$

Conversely, for a pandemic that comes to an end:

$$\exists t_1 > 0 \mid \begin{array}{ll} \dot{y}(t) > 0, & \text{se } 0 \leq t < t_1 \\ \dot{y}(t) = 0, & \text{se } t \geq t_1 \end{array} \tag{31}$$

But all the derivatives at the point t_1. must cancel out. For example, if $\ddot{y}(t_1) \neq 0$, the graph of $\dot{y}(t)$ has an angular point in $(t_1, 0)$ and this behavior is not predicted by (30). Iterating:

$$\left. \frac{dy}{dt} \right|_{t=t_1} = 0, \quad \left. \frac{d^2 y}{dt^2} \right|_{t=t_1} = 0, ..., \left. \frac{d^n y}{dt^n} \right|_{t=t_1} = 0, ...$$

In other words, derivatives of a high order must be canceled out. It follows the not analyticity of the aforesaid function. On the other hand, for an assigned analytical function $\beta(t)$, the solution of the Cauchy problem (24) is in turn an analytic function. We have so proved the theorem:

Theorem 12 *A pandemic $\mathcal{P}$ modeled by a non-autonomous system of type (24) at most dies out asymptotically.*

In practice, in the case of asymptotic extinction we refer to a stop instant t_* such as to verify a condition of type (23).

2.4 Configuration domain

Let us return to the special case of an autonomous system (13).

Definition 13 *The following subset of* $\mathbb{R}^2$

$$\{(y,\dot{y}) \mid 0 \leq y < +\infty, \ -\infty < \dot{y} < +\infty\}$$

it is called **the configuration space** *of the dynamic system (13).*

Definition 14 *The geometric locus (15) è is the region of the configuration space* **accessible** *to the dynamic system (13). The generic point* $(y,\dot{y}) \in \Gamma(f)$ *is called* **the representative point** *of the system.*

In the particular case of exponential growth, the region of the configuration space accessible to this system is the line of equation $\dot{y} = R_0 y$, i.e. the line for the origin and the angular coefficient $R_0 > 0$.

The notion of configuration space and the corresponding region accessible to the system suggests an alternative approach to the study of the dynamic evolution of an autonomous system. Precisely, instead of integrating the differential equation

$$\dot{y} = f(y) \tag{32}$$

for an assigned initial condition, the evolution of the representative point in the configuration space is studied.

2.4.1 Sampling

The analysis just seen is valid for any autonomous system characterized by a quantity $y(t)$ which is a real function of the real variable t. However from § 1 it follows that y, t are not variable quantities with continuity. For the independent variable we write:

$$t_k = k\Delta, \qquad k = 0, 1, 2, ..., n, \tag{33}$$

where $\Delta = 1\,\mathrm{d}$. On the other hand, remaining continuously (14) we write:

$$\lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t} = f[y(t)],$$

while the sampling (33) of the independent variable uniquely determines the sampling of the dependent variable, so

$$\frac{y_{k+1} - y_k}{\Delta} = f(y_k),$$

having defined $y_k = y(t_k)$. It follows

$$y_{k+1} = y_k + f(y_k)\Delta,$$

that is, an equation of recurrence for y. The representative point of the system therefore moves by discrete steps along the curve of the cartesian plane (y_k, y_{k+1}):

$$\gamma : y_{k+1} = g_\Delta(y_k), \tag{34}$$

where is it

$$g_\Delta(y_k) \stackrel{def}{=} y_k + f(y_k)\Delta \tag{35}$$

The set of positions taken by the aforementioned representative point makes up the *diagram of the orbits* of the system whose *transfer function* is (35).

This computing paradigm has produced a vast literature opening new and fantastic horizons. Search for fixed points, ergodicity, chaos (i.e. deterministic chaos) are the elements that distinguish these systems [3].

3 Random variables

In a realistic scenario, random behavior is expected for what concerns the containment action. This suggests to model $\beta(t)$ through a suitable random variable. Suppose:

$$\beta(t) = \beta_0 + \beta_1(t), \quad |\beta_1(t)| \ll \beta_0$$

where $\beta_1(t)$ is a random variable. To be more precise, $\beta_1(t)$ is an stationary random process and its random fluctuations model deviations from *correct behavior* (social distancing, use of masks, etc.). In other words, we have a constant containment parameter β_0 immersed in a "noise".

3.1 Wiener Processes

Let us consider in particular, a *Wiener process* i.e. the integral of a white noise $W(t)$. As is known, the latter has a flat power spectrum:

$$w(f) = w_0 > 0, \ \forall f \in \mathbb{R}$$

where f is the frequency. From the Wiener–Khintchine Theorem it follows that the autocorrelation function is deltiform:

$$\varphi(\tau) = w_0 \delta(\tau), \ \forall \tau \in \mathbb{R}$$

In other words, the values assumed by $W(t)$ are 100% uncorrelated. For the above, a Wiener process (also known as *Brown noise*) is an integral of $W(t)$. With abuse of notation, we write:

$$\beta_1(t) = \int W(t)\, dt \tag{36}$$

The abuse derives from the fact that the quantity $W(t)$ is not a function in the sense of mathematical analysis. However, the Mathematica software ooffers the possibility to manipulate these objects in the same way as the usual functions. Specifically, after generating an array of values assumed by $W(t)$ for an assigned range of values assumed by t, we use the instruction `Interpolation[]` to create a real function of the real variable t, symbolized by

$$W_{int}[t]$$

after which we calculate its primitive through the usual `Integrate[]`instruction. In symbols:

$$W_{int}[t] \underset{\int}{\longrightarrow} \beta_1[t]$$

The result is reported in fig. 5.

From the point of view of mathematical analysis, a white noise is a "function" that has a point of discontinuity of the first kind (or *finite discontinuity*) in every point of its field of existence. Furthermore, the assumed values are random. For the above, *Mathematica* generates a random list of the aforementioned values. With the `Interpolation[]` instruction, *Mathematica* performs a polynomial interpolation. In this way a function with very violent oscillations is obtained but still integrable in the sense of Riemann. The presence of an infinite number of discontinuities of the first kind implies that its primitive is a function whose graph has an equally infinite number of angular points, just as shown in fig. 5. With an appropriate choice of the parameters that enter the differential equation gives the result of 6, which qualitatively reflects the Italian situation in today's period.

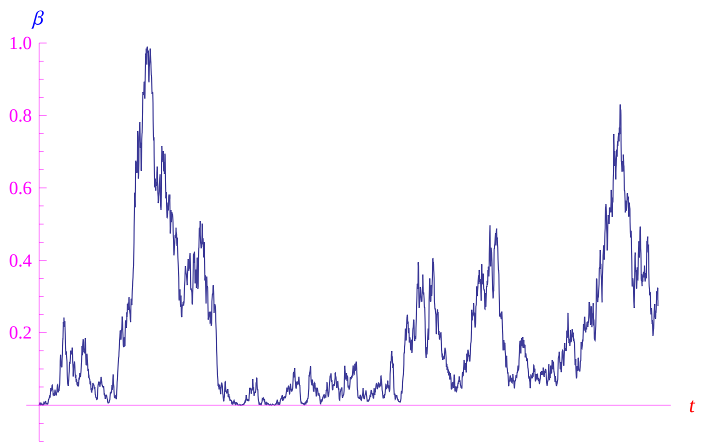

Figure 5: Typical trend of a Wiener process.

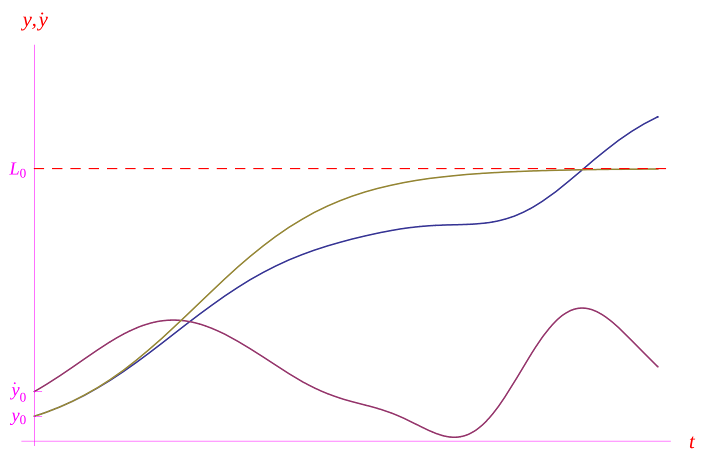

Figure 6: The blue curve is an integral curve of the Cauchy problem (24), where $\beta(t)$ is a Wiener process.

A Density fluctuations in the number of infected

A.1 Introduction

These notes follow from a development of the article by Davide Lombardi

A.2 Balance equation

Let $I(t)$ be the number of infected individuals at time t. We assume that this quantity is a deterministic variable

$$I(t) \underset{\text{ev. deterministica}}{\longrightarrow} I(t'), \quad \forall t' > t$$

That is, the value assumed by I at a given instant t univocally determines the value assumed at any future instant. It follows that the function $I(t)$ is the only solution of a Cauchy problem of the type:

$$\begin{cases} \dot{I} = F(t, I) \\ I(t_0) = 0 \end{cases} \tag{37}$$

Obviously we are interested in the elementary expression of the function $F(t, I)$ in order to be able to solve the aforementioned problem. For this purpose, we take at will a limited region of physical space, mathematically represented by a regular and limited domain D whose boundary ∂D is a regular surface. We denote by $I_D(t)$ the restriction of the function $I(t)$ to the aforementioned domain, which can be expressed through a density function $i(x, t)$ which returns the number of infected individuals (simply infected) at time t and in the unit of volume.

$$I_D(t) = \int_D i(\mathbf{x}, t)\, d^3x \tag{38}$$

Deriving

$$\frac{d}{dt} I_D(t) = \frac{d}{dt} \int_D i(\mathbf{x}, t)\, d^3x \tag{39}$$

Equation (39) measures the rate of change of $I_D(t)$ in D, i.e. the variation in the unit of time (in D) of the number of infected. In turn, this variation is the algebraic sum of two contributions:

1. number of infected people who enter or leave D through its border in the unit of time;
2. number of new infected (in D) in the unit time interval.

Contribution 1 assumes the existence of a velocity field in D, with which the infected move, and which we denote by $\mathbf{v}(\mathbf{x}, t)$. If $d\boldsymbol{\sigma} = \mathbf{n} d\sigma$ is the surface element oriented according to the unit vector $\mathbf{n}$ the normal external to ∂D (fig. 7), the number of infected in the unit of time cross $d\sigma$ is given by $\mathbf{j}(\mathbf{x}, t) \cdot d\boldsymbol{\sigma}$, dove $\mathbf{x}$ è il vettore posizione del punto di ∂D in cui valutiamo $d\sigma$, while the quantity

$$\mathbf{j}(\mathbf{x}, t) = i(\mathbf{x}, t)\, \mathbf{v}(\mathbf{x}, t) \tag{40}$$

is the current density of the infected. It follows that the contribution 1 is

$$\oint_{\partial D} \mathbf{j}(\mathbf{x}, t) \cdot \mathbf{n} d\sigma = \Phi_{\partial D}(\mathbf{j}) \tag{41}$$

that is the flow of the vector $\mathbf{j}$ through ∂D. Now suppose that contribution 2 is zero, that is, in D no new infected are created. Necessarily

$$\left| \frac{d}{dt} \int_D i(\mathbf{x}, t)\, d^3x \right| = |\Phi_{\partial D}(\mathbf{j})|$$

It follows

$$\frac{d}{dt} \int_D i(\mathbf{x}, t)\, d^3x > 0 \Longrightarrow I_D(t) \text{ è crescente}$$
$$\Longrightarrow \text{entrano nuovi infetti} \Longrightarrow \Phi_{\partial D}(\mathbf{j}) < 0$$

and viceversa. Therefore

$$\frac{d}{dt} \int_D i(\mathbf{x}, t)\, d^3x = -\Phi_{\partial D}(\mathbf{j}) \tag{42}$$

The contribution 2 is given by a quantity $\Gamma_D(t)$ equal to the speed with which the number of infected in D increases. Let us express it through a density function $\gamma(\mathbf{x}, t)$ or the speed per unit of volume with which the number of infected grows.

$$\Gamma_D(t) = \int_D \gamma(\mathbf{x}, t)\, d^3x \tag{43}$$

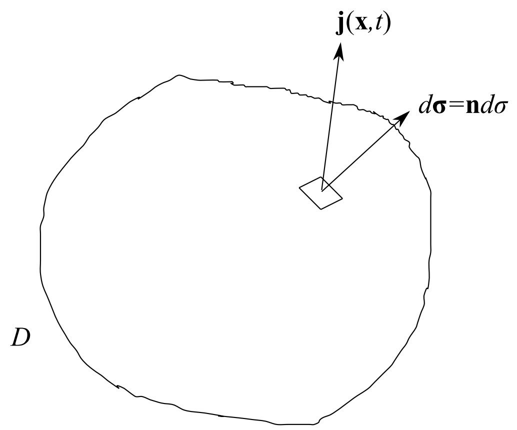

Figure 7: By D we denote a regular domain of $\mathbb{R}^3$ representative of an arbitrary region of physical space.

We finally have the *balance equation*:

$$\frac{d}{dt} \int_D i(\mathbf{x}, t)\, d^3x = -\Phi_{\partial D}(\mathbf{j}) + \int_D \gamma(\mathbf{x}, t)\, d^3x, \quad \forall D \subset \mathbb{R}^3 \tag{44}$$

For $D = \mathbb{R}^3$

$$\frac{d}{dt} \int_{\mathbb{R}^3} i(\mathbf{x}, t)\, d^3x = \int_{\mathbb{R}^3} \gamma(\mathbf{x}, t)\, d^3x \tag{45}$$

since $\Phi_{\partial \mathbb{R}^3}(\mathbf{j}) = 0$. Formally, this result is reached by assuming a cube with edge L as a domain, and then performing the operation of passing to the limit for $L \to +\infty$.

A.3 Statistical analysis

Now let's calculate the spatial mean:

$$\langle i\rangle_L(t) = \frac{1}{L^3}\int_D i(\mathbf{x},t)\,d^3x, \tag{46}$$

and therefore

$$\langle i\rangle(t) = \lim_{L\to+\infty}\frac{1}{L^3}\int_D i(\mathbf{x},t)\,d^3x \tag{47}$$

The quantity $i(\mathbf{x},t)$ can be treated as a random variable, of variance space (or *space power* of quantity $i(\mathbf{x},t)$):

$$\sigma^2(t) = \left\langle [i(\mathbf{x},t) - \langle i\rangle(t)]^2\right\rangle = \left\langle i^2\right\rangle(t) - \langle i\rangle^2(t), \tag{48}$$

where the spatial mean is computed according to (47). We decompose the scalar field $i(\mathbf{x},t)$ in a superposition of plane waves, imposing periodic conditions on the faces of the cube. Precisely, we develop in Fourier series:

$$i(\mathbf{x},t) = \sum_{\mathbf{k}} i_{\mathbf{k}}(t)\,e^{j\mathbf{k}\cdot\mathbf{r}}, \quad (j=\sqrt{-1})$$

whose Fourier coefficients in the space of the wave vectors $\mathbf{k}$, are

$$i_{\mathbf{k}}(t) = \frac{1}{L^3}\int_D i(\mathbf{x},t)\,e^{-j\mathbf{k}\cdot\mathbf{r}}d^3r \tag{49}$$

By imposing the reality of $i(\mathbf{x},t)$:

$$i(\mathbf{x},t) \equiv i^*(\mathbf{x},t) \Longleftrightarrow i^*_{\mathbf{k}}(t) \equiv i_{-\mathbf{k}}(t) \tag{50}$$

The periodic conditions are

$$i(0,y,z) = i(L,y,z), \quad \text{e simili}$$

which give rise in the space of the wave numbers under the conditions:

$$k_x = \frac{2\pi}{L}n_x, \quad k_y = \frac{2\pi}{L}n_y, \quad k_z = \frac{2\pi}{L}n_z, \qquad n_x, n_y, n_z \in \mathbb{Z}$$

For $L\to+\infty$ the Fourier series is a Fourier transform:

$$i(\mathbf{x},t) = \int_{\mathbb{R}^3} i(\mathbf{k},t)\,e^{j\mathbf{k}\cdot\mathbf{r}}d^3k \tag{51}$$
$$i(\mathbf{k},t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} i(\mathbf{x},t)\,e^{-j\mathbf{k}\cdot\mathbf{r}}d^3r$$

In conditions of isotropy in the space of wave numbers:

$$i(\mathbf{x},t) = 4\pi\int_0^{+\infty} i(k,t)\,k^2e^{j\mathbf{k}\cdot\mathbf{r}}dk$$
$$i(k,t) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} i(\mathbf{x},t)\,e^{-j\mathbf{k}\cdot\mathbf{r}}d^3r$$

This implies that potency is expressed as:

$$\sigma^2(t) = 4\pi \int_0^{+\infty} w(k,t)\, k^2 dk,$$

where $w(k,t)$ is the *spatial power spectrum* of quantity $i(\mathbf{x},t)$. The autocorrelation function of $i(\mathbf{x},t)$ is

$$\psi(\mathbf{r},t) = \langle i(\mathbf{x},t)\, i(\mathbf{x}+\mathbf{r},t)\rangle \tag{52}$$

By the Wiener–Khintchine Theorem

$$w(k,t) = 4\int_{\mathbb{R}^3} \psi(\mathbf{r},t)\, e^{j\mathbf{k}\cdot\mathbf{r}} d^3r$$
$$\psi(\mathbf{r},t) = \int_0^{+\infty} w(k,t)\, k^2 e^{-j\mathbf{k}\cdot\mathbf{r}} dk$$

We rewrite (45) as:

$$\frac{d}{dt} \lim_{L\to+\infty} \int_{D_L} i(\mathbf{x},t)\, d^3x = \lim_{L\to+\infty} \int_{D_L} \gamma(\mathbf{x},t)\, d^3x \tag{53}$$

Before carrying out the operation of passing to the limit we can multiply both members for L^{-1}

$$\underbrace{\frac{d}{dt} \lim_{L\to+\infty} \frac{1}{L} \int_{D_L} i(\mathbf{x},t)\, d^3x}_{=\langle i\rangle(t)} = \underbrace{\lim_{L\to+\infty} \frac{1}{L} \int_{D_L} \gamma(\mathbf{x},t)\, d^3x}_{\langle\gamma\rangle(t)}$$

That is

$$\frac{d}{dt}\langle i\rangle(t) = \langle\gamma\rangle(t) \tag{54}$$

To make the second member explicit, we expect the velocity density $\gamma(\mathbf{x},t)$ to be a composite function of the type

$$\gamma(\mathbf{x},t) = f[i(\mathbf{x},t)]$$

By developing the function $f(i)$ in a right neighborhood of $i=0$ in Taylor series

$$f(i) = c_0 + c_1 i + c_2 i^2 + \dots$$

We observe that if the density $i(\mathbf{x},t)$ is identically zero, such will be $\gamma(\mathbf{x},t)$, so

$$c_0 = f(0) = 0$$

So, truncating to the second order:

$$f(i) = c_1 i + c_2 i^2 \Longrightarrow \langle\gamma\rangle(t) = \left\langle c_1 i + c_2 i^2\right\rangle(t) = c_1\langle i\rangle(t) + c_2\left\langle i^2\right\rangle(t) \tag{55}$$

From (48):

$$\left\langle i^2\right\rangle(t) = \langle i\rangle^2(t) + \sigma^2(t)$$

Replacing in the previous one

$$\langle\gamma\rangle(t) = c_1\langle i\rangle(t) + c_2\langle i\rangle^2(t) + c_2\sigma^2(t)$$

And therefore in (54)

$$\frac{d\langle i\rangle}{dt}=c_1\langle i\rangle+c_2\langle i\rangle^2+c_2\sigma^2(t) \tag{56}$$

which is the differential equation (37) referred to the average density. It is a non-linear and non-autonomous first order differential equation, due to the time dependence of the power spatial $\sigma^2(t)$. The effects of this last quantity depend on the sign of the coefficient c_2 of the Taylor expansion and therefore, of the second derivative $f''(0)$. Precisely:

$$\frac{d\langle i\rangle}{dt}=\begin{cases} c_1\langle i\rangle+c_2\langle i\rangle^2+c_2\sigma^2(t), & \text{se } c_2>0 \\ c_1\langle i\rangle-|c_2|\langle i\rangle^2-|c_2|\sigma^2(t), & \text{se } c_2<0\end{cases} \tag{57}$$

In other words, if $c_2>0$ the power of $i(\mathbf{x},t)$ amplifies the viral growth. Conversely, if $c_2<0$ the power acts as a damping factor, since the derivative of the mean value of $\langle i\rangle$ is reduced by a factor $c_2\sigma^2(t)$. For $c_2<0$, if the distribution is homogeneous i.e. $\sigma^2(t)\equiv 0$, has the classic *logistic evolution*:

$$\frac{d\langle i\rangle}{dt}=c_1\langle i\rangle-|c_2|\langle i\rangle^2 \tag{58}$$

From the W-K theorem it follows that to have a null power spectrum, there must be the maximum correlation between the values assumed by the scalar field $i(\mathbf{x},t)$ at the various points $\mathbf{x}$. Conversely, if the random variable $i(\mathbf{x},t)$ is a *white noise*, i.e. its power spectrum is flat:

$$\sigma^2(t)=4\pi\int_0^{k_{\max}}k^2dk=\frac{4}{3}\pi k_{\max}^3\equiv\sigma_{\max}$$

and the values assumed by the field are uncorrelated to 100%, i.e. the autocorrelation function as a Fourier transform of a flat power spectrum, is a three-dimensional Dirac delta centered at $\mathbf{r}=0$:

$$\psi(\mathbf{r})=\delta^{(3)}(\mathbf{r})$$

In this case, (57) becomes (if $c_2<0$)

$$\frac{d\langle i\rangle}{dt}=c_1\langle i\rangle-|c_2|\langle i\rangle^2-|c_2|\sigma_{\max} \tag{59}$$

to which corresponds the minimum value of the speed of variation of the average density of the number of infected ($\frac{d\langle i\rangle}{dt}$). For the above, in the limit $\sigma(t)\to 0$, the distribution of infected is homogeneous. Interpreting the viral diffusion as the propagation of a signal between individuals (whose motion can be described by the propagation of a Dirac delta wave), we have that in the homogeneous case the propagation of the aforementioned signal is instantaneous, for which there will be a single cluster that expands uniformly (following the logistic trend 58). Conversely, the condition $\sigma>0$ makes the distribution inhomogeneous and this determines the presence of disjoint clusters where the density of the infected is governed by the non-homogeneous differential equation (59).

Notation 15 *We note incidentally that this conclusion exhibits a remarkable (albeit formal) analogy with Friedmann's cosmological models. More precisely, the universe primordial described by these models is homogeneous (Milne's Principle): the density of matter-energy does not depend on the spatial coordinates (in co-moving) but only on the time coordinate. In this way the expansion of the universe occurs in the Hubble flow, conserving the initial homogeneity. But such a universe does not produce gravitational structures such as galaxies* *and clusters of galaxies. In a more realistic description, fluctuations are considered of density (triggered by quantum processes) which then grew (Jeans theory) to then exit the Hubble flow giving rise to the aforementioned structures. From a statistical point of view, the distribution of galaxies in the current universe is studied through the two-point correlation function (or more generally, n-points). In the case of a pandemic instead of the density of matter-energy we have the density of infected. A homogeneous distribution does not produce structures of the disjoint cluster type, but there is a single expanding cluster. As in the cosmological case, here too the initial homogeneity is preserved (curiously, even in Cosmology there is the problem of the instantaneous propagation of a "signal", since regions outside their respective cosmological horizons were thermalized). Conversely, the presence of fluctuations in the density of infected destroys the homogeneity by creating disjoint clusters that they do not communicate instantly. The n-point correlation function could be of some use to explain some anomalies of viral propagation that occurred in Italy in the period February-June (maximum density in the northern regions, minimum in the south).*

A.4 The two-point correlation function. Fractal distribution

In a nutshell: initially, the viral spread follows an exponential law until containment actions take place, after which the process tends to follow the classic logistics. For the above, this occurs if and only if there is the greatest correlation between infected individuals. Conversely, in the presence of fluctuations in the density of infected, the cluster tends to fragment. In symbols:

$$\text{cluster iniziale} \underset{\text{fragmentation}}{\longrightarrow} N \gg 1 \text{ clusters}$$

Assigned a cartesian reference $\mathcal{R}\,(Oxyz)$ with origin in the center of the representative sphere Σ della Terra, of the Earth, and with axes xyz so as to compose a three-rectangle left-handed, we denote by $\rho\,(\mathbf{r},t)$ a non-negative function such that

$$dP = \rho\,(\mathbf{r}_0, t)\, dV \tag{60}$$

is the infinitesimal probability of finding, at instant t, a cluster in the volume element $dV = dxdydz$ centered in the vector point position $\mathbf{r}_0 = (x_0, y_0, z_0)$, for which the aforesaid function is a probability density that verifies l'obvious condition of normalization:

$$\int_{\mathbb{R}^3} \rho\,(\mathbf{r}, t)\, dV = 1, \quad \forall t \in [0, +\infty) \tag{61}$$

Orienting the z-axis of the cartesian reference $\mathcal{R}$ in the direction and towards the north pole of the earth, we pass from the Cartesian coordinates to the polar coordinates in space (*spherical coordinates*) spherical coordinates O of $\mathcal{R}$ and the polar axis coinciding with the z-axis. The equations that connect the cartesian coordinates (x, y, z) to the spherical coordinates (r, θ, φ) are:

$$x = R_T \sin\theta \cos\varphi, \;\; y = R_T \sin\theta \sin\varphi, \;\; z = R_T \cos\theta, \tag{62}$$

since for any point of Σ the radial coordinate is $r = R_T$, where the latter is the radius of the earth. Conventionally, we assume R_T as a unit of length, i.e. we set $R_T \equiv 1$. Recall that the angular coordinates θ, φ are respectively called *colatitude* and *longitude*:

$$0 \leq \theta \leq \pi, \;\; 0 \leq \varphi \leq 2\pi$$

and they are obviously linked to the geographic coordinates latitude and longitude, even if there is a conflict of symbols since the longitude (as a geographic coordinate) is indicated with λ and is measured in degrees from 0 a 180° towards East or West. The latitude is instead symbolized by φ, while the θ colatitude of the spherical coordinates is the complementary angle.

In spherical coordinates the volume element appearing in (60) is written:

$$dV = r^2 dr d\Omega, \tag{63}$$

where $d\Omega$ is the elementary solid angle:

$$d\Omega = \sin\theta d\theta d\varphi \tag{64}$$

Since clusters are portions of Σ, in (60) the volume element must be replaced by the surface element of Σ:

$$dS = R_T^2 d\Omega \underset{R_T=1}{=} d\Omega \tag{65}$$

Therefore

$$dP = \rho(\mathbf{r}_0, t)\, d\Omega \tag{66}$$

Here the components of the vector $\mathbf{r}_0$ are expressed according to (62) with $R_T = 1$.

That said, the joint probability of finding cluster 1 in $d\Omega_1$ centered in $\mathbf{r}_1$ at instant t, and cluster 2 in $d\Omega_2$ centered in $\mathbf{r}_2$, is

$$d^2P = dP_1 dP_2,$$

being

$$dP_1 = \rho(\mathbf{r}_1, t)\, d\Omega_1, \;\; dP_2 = \rho(\mathbf{r}_2, t)\, d\Omega_2$$

So

$$d^2P = \rho(\mathbf{r}_1, t)\, \rho(\mathbf{r}_2, t)\, d\Omega_1 d\Omega_2 \tag{67}$$

Equation (67) is valid if the relative positions of the individual clusters are uncorrelated. Otherwise, this equation becomes:

$$d^2P = \rho(\mathbf{r}_1, t)\, \rho(\mathbf{r}_2, t)\, d\Omega_1 d\Omega_2 + d^2P_*$$

where d^2P_* is the excess or defect in probability, which can be expressed as:

$$d^2P_* = \xi_c(\mathbf{r}_{12}, t)\, \rho(\mathbf{r}_1, t)\, \rho(\mathbf{r}_2, t)\, d\Omega_1 d\Omega_2 \tag{68}$$

Here $\mathbf{r}_{12} = r_1 - r_2$ is the relative position vector of the clusters, while $\xi_c(\mathbf{r}_{12}, t) \geq 0$ is a dimensionless quantity known as a **two-point spatial correlation function** or simply a *two-point correlation function*.

In other words, if $\xi_c(\mathbf{r}_{12}, t) \equiv 0$ there is a random distribution of clusters, while if $\xi_c(\mathbf{r}_{12}, t) > 0$ lthe distribution tends to form structures or leave gaps. We therefore have:

$$d^2P = \rho(\mathbf{r}_1, t)\, \rho(\mathbf{r}_2, t)\, [1 + \xi_c(\mathbf{r}_{12}, t)]\, d\Omega_1 d\Omega_2 \tag{69}$$

These arguments are generalized by defining a correlation function at $q > 2$ points, for which the joint probability of finding at instant t, q clusters in $\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_q$ respectively, is

$$d^qP = \rho(\mathbf{r}_1, t)\, \rho(\mathbf{r}_2, t) ... \rho(\mathbf{r}_q, t)\, [1 + \xi_c(\mathbf{r}_{12}, \mathbf{r}_{13}, \mathbf{r}_{23}, ..., t)]\, d\Omega_1 d\Omega_2 ... d\Omega_q \tag{70}$$

Without loss of generality, we consider the case $q = 2$. The vector defining the relative position is

$$\mathbf{r}_{12} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

In spherical coordinates:

$$\mathbf{r}_{12} = (\sin\theta_2\cos\varphi_2 - \sin\theta_1\cos\varphi_1, \sin\theta_2\sin\varphi_2 - \sin\theta_1\sin\varphi_1, \cos\theta_2 - \cos\theta_1)$$

It follows that the correlation function ξ_c becomes a function composed of the variables $\theta_1, \varphi_1, \theta_2, \varphi_2$. In order not to weigh down the notation, we continue to indicate this function with the same symbol:

$$\xi_c(\mathbf{r}_{12}, t) = \xi_c(\theta_1, \varphi_1, \theta_2, \varphi_2, t) \tag{71}$$

Since the orientation of the polar axis is arbitrary, we orient this axis in the direction of cluster 1, so that in the new spherical coordinate system this cluster has theta colatitude $\theta = 0$ and indeterminate longitude φ. Conventionally we set $\varphi = 0$. So in (71):

$$\theta_1 = 0, \varphi_1 = 0$$

Therefore, the independent variables θ_2, φ_2, t remain, and we redefine the angular ones simply with θ, φ. It follows

$$\xi_c(\theta, \varphi, t), \quad \forall(\theta, \varphi) \in [0, \pi] \times [0, 2\pi], \quad \forall t \in [t_0, +\infty) \tag{72}$$

Moreover, in the limit of large N, we can approximate the probability density with the average number at instant t of clusters in the unit of surface:

$$\rho(\mathbf{r}, t) \sim \langle n\rangle(t) \tag{73}$$

Thus (69) becomes:

$$dP = \langle n\rangle(t)\left[1 + \xi_c(\theta, \varphi, t)\right] d\Omega \tag{74}$$

or what is the same

$$dP = \rho_c(\theta, \varphi, t)\, d\Omega$$

having defined the probability density of finding a cluster in (θ, φ) at instant t

$$\rho_c(\theta, \varphi, t) = \langle n\rangle(t)\left[1 + \xi_c(\theta, \varphi, t)\right] \tag{75}$$

which verifies the normalization condition:

$$\int_{4\pi} \rho_c(\theta, \varphi, t)\, d\Omega = 1,$$

having extended the integration to the total solid angle. That is

$$\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi \rho_c(\theta, \varphi, t) = 1$$

From (75) it follows that the correlation function $\xi_c(\theta, \varphi, t)$ verifies the normalization condition:

$$\int_{4\pi} \xi_c(\theta, \varphi, t)\, d\Omega = \frac{1 - 4\pi\langle n\rangle(t)}{\langle n\rangle(t)}, \quad \forall t \in [t_0, +\infty) \tag{76}$$

Assigned an initial condition

$$\xi_c(\theta, \varphi, t_0) \equiv \xi_c^{(0)}(\theta, \varphi) \tag{77}$$

the problem consists of determining the time evolution of the initial configuration. That is:

$$\xi_c^{(0)}(\theta, \varphi) \underset{\text{ev. temporale}}{\longrightarrow} \xi_c(\theta, \varphi, t) \tag{78}$$

Since initially there is only one cluster:

$$\xi_c^{(0)}(\theta, \varphi) = \delta(\theta)\,\delta(\varphi),$$

where δ denotes the Dirac delta function. The problem thus arises of determining the dynamic evolution of the correlation function starting from an initial deltiform configuration. It could be conjectured that, with a statistical distribution described by a correlation function like:

$$\xi_c(\theta, \varphi, t) \propto \theta^{\mu}\varphi^{\nu}, \quad \theta \in (0, \Delta\theta), \ \varphi \in (0, \Delta\varphi), \ \forall t > 0$$

with $1 < \mu, \nu < 2$, the average number of clusters in the region

$$T = \{(r, \theta, \varphi) \mid r = R, \ 0 < \theta < \Delta\theta, \ 0 < \varphi < \Delta\varphi\}$$

is

$$n \propto l^D \tag{79}$$

where l is the distance (on the sphere) between the initial cluster (in $(\theta, \varphi) = (0, 0)$) and ∂T. In (79) $D > 0$ is a fractional exponent called the *fractal dimension* of the distribution of clusters.

B Bernoulli equation

The ordinary nonlinear first-order differential equation:

$$\dot{y} = R_0 y - \beta(t)\, y^2 \tag{80}$$

of the Cauchy problem (24) is a Bernoulli equation and is integrated by setting:

$$u = \frac{1}{y}, \tag{81}$$

whereby (80) becomes

$$\dot{u} + R_0 u = \beta(t), \tag{82}$$

which is a non-homogeneous linear first-order differential equation. An integral factor is $e^{R_0 t}$, so multiplying the first and second members of (80) by this factor, we have:

$$\frac{d}{dt}\left[u(t)\, e^{R_0 t}\right] = \beta(t)\, e^{R_0 t}$$

By integrating first and second members with respect to t

$$u(t, C)\, e^{R_0 t} = C + \int \beta(t)\, e^{R_0 t} dt, \quad \forall C \in \mathbb{R} \tag{83}$$

where C is an integration constant. It follows

$$u(t,C) = Ce^{-R_0 t} + e^{-R_0 t} \int \beta(t) e^{R_0 t} dt \tag{84}$$

From (81):

$$y(t,C) = \frac{e^{R_0 t}}{C + \int \beta(t) e^{R_0 t} dt} \tag{85}$$

By imposing the initial condition $y(0,C) = y_0$, we obtain

$$y(t) = \frac{y_0 e^{R_0 t}}{1 - y_0 [B_0 - B(t)]} \tag{86}$$

where

$$B(t) \overset{def}{=} \int \beta(t) e^{R_0 t} dt, \quad B_0 = B(0) \tag{87}$$

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